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RESEARCH ARTICLE

Periodical Solutions for Extended Kalman Filter's Stability

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Abstract---In this paper, we study the global stability, asymptotic properties of the nonnegative solutions and periodic solutionsof the nonlinear rational difference equation

$$x_{n+1} = Ax_n + Bx_{n-1} + \frac{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}{A + Bx_{n-2} + Cx_{n-3}}, n = 0, 1, 2, \ . \ . \ .$$

where the non negative parameters $A, B, C, 1\alpha, \beta, \gamma$and the arbitrary non negative initial conditions $x_{-k}, x_{-(k-1)} \dots, x_{-1}, x_0 \in (0, \infty)$ are arbitrary nonnegative real numbers and kis a positive integer number. Also we analyze the bounded characters of the rational difference equation. Finally some numerical examples are studied and draw by Mathematica 8.0.4.

Mathematics Subject Classifications---39A10, 39A11, 39A99, 34C99

Keywords---Equilibrium, asymptotic, positive solution, rational difference equation, boundedness character, stability.

I. INTRODUCTION

The study of difference equations has been growing continuously in the last decade. Difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, genetics in biology, economics, psychology, sociology, digital signal processing , filter theory etc. So, recently, there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Many researchers are interested in the boundedness, invariant intervals, periodic character and global asymptotic stability of positive solutions for nonlinear difference equations, [2 -6]. Periodic solutions of difference equations have been investigated by many researchers, and various methods have been proposed for the existence and qualitative properties of the solution. The study of rational difference equations of order greater than one is quite challenging and rewarding. There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations.Our goal in this paper is to investigate some qualitative behavior of the solutions of the nonlinearrational difference equation

$$x_{n+1} = Ax_n + Bx_{n-1} + \frac{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}{A + Bx_{n-2} + Cx_{n-3}} \qquad (1)$$

$$n = 0,1,2, \ . \ . \ .$$

Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear rational difference equations. The paper proceeds as follows. In Section 2, we recall some preliminary discussion and lemmas.Section 3 describes the periodic solutions of (1). Local and global stability of the equilibrium point of (1) is arrived in Section 4. Boundedness characters of the positive solutions of (1) are discussed in Section 5. The results of Section 3 to 5 is implemented in extended Kalman filter and its stability, which are described in Section 6. Finally we give numerical examples of some special cases of (1) and draw it by using Mathematica. Section 7 concludes the paper.

II. PRELIMINARIES

*Definition2. 1.*A difference equation of order $(k\ +1)$is of the form

$$x_{n+1} = F(x_n, x_{n-k}), \qquad n = 0,1,2, \dots \qquad (2)$$

where F is a continuous function which maps some set $\mathbb{R}^{k+1}$into $\mathbb{R}$ where $\mathbb{R}$ is a set of real numbers. An equilibrium point $\bar{x}$of this equation is a point that satisfies the condition $\bar{x} = F(\bar{x}, \bar{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$with $x_n = \bar{x}$ for all $n \geq -k$ is a solution ofthat equation.

*Definition 2.2*Let$\bar{x} \in (0,\infty)$be an equilibrium point of the difference equation (2). Then,

(i)An equilibrium point $\bar{x}$of the equation (2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution to (2) with the property that$|x_{-k}-\bar{x}|+|x_{-k+1}-\bar{x}|+ \ldots + +|x_0-\bar{x}|<\delta$, then $|x_n-\bar{x}|<\varepsilon$ for all $n \geq -k$ where$x_{-k}, x_{-(k-1)} \ldots, x_{-1}, x_0 \in (0,\infty)$.

(ii)An equilibrium point $\bar{x}$of theequation (2) is called locally asymptotically stable if $\bar{x}$ is locally stable, and if there exists a $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution to (2) with the property

$|x_{-k}-\bar{x}|+|x_{-k+1}-\bar{x}|+ \ldots +|x_0-\bar{x}|<\gamma$, then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii)An equilibrium point $\bar{x}$of the equation (2) is called a global attractor if for every solution, $\{x_n\}_{n=-k}^{\infty}$ of (2), we have

$$\lim_{n\to\infty} x_n = \bar{x}$$

where $x_{-k}, x_{-(k-1)} \ldots, x_{-1}, x_0 \in (0,\infty)$.

(iv)An equilibrium point $\bar{x}$of the equation (2) is called globally asymptotically stableif it is locally stable and a global attractor.

(v)An equilibrium point $\bar{x}$ of the difference equation (2) is called unstable if it is not locally stable.

Definition 2.3(2) is said to be permanent if there exist positive real numbers m and M such that for every solution$\{x_n\}_{n=-k}^{\infty}$ of (2) there exists a positive integer $N \geq -k$which depends on the initial conditions, such that

$$m \leq x_n \leq M \text{ for all } n \geq N.$$

The linearized equation of the difference equation (2) about the equilibrium point $\bar{x}$is also the linear difference equation which is

$$z_{n+1} = \frac{\partial F(\bar{x},\bar{x})}{\partial x_n} z_n + \frac{\partial F(\bar{x},\bar{x})}{\partial x_{n-k}} z_{n-k} \qquad (3)$$

The characteristic equation associated with (3) is

$$p(\lambda) = \lambda^{k+1} - p_0\lambda^k - p_1 = 0 \qquad (4)$$

Where

$$p_0 = \frac{\partial F(\bar{x},\bar{x})}{\partial x_n}, p_1 = \frac{\partial F(\bar{x},\bar{x})}{\partial x_{n-k}}$$

*Theorem 2.4. The linearized stability theorem.*Suppose F is a continuously differentiable function defined on an open neighbourhood of the equilibrium $\bar{x}$. Then the following statements are true.

(i)If all the roots of the characteristic equation (4) of the linearized equation (3) have absolute value less than one, then theequilibrium point $\bar{x}$ of (2) is locally asymptotically stable.

(ii)If at least one root of (4) has absolute value greater than one, then the equilibrium point $\bar{x}$ of (2).

(iii) If all the roots of (4) have absolute value greater than one, then the equilibrium point $\bar{x}$ of (2).

*Theorem 2.5*Assume that $p_i \in \mathbb{R}, i = 1,2, \ldots$and $k \in \{0,1,2,\ldots\}$. Then

$$\sum_{i=}^{k} |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0 , n = 0,1,2, \ldots$$

The following global attractively result is very useful in establishing convergence results in many situations.

*Theorem 2.6*Let $g: [\alpha,\beta]^{k+1} \to [\alpha,\beta]$be a continuous function, where k is a positive integer and where $[\alpha,\beta]$is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-k}), \qquad n = 0,1, \ldots \qquad (5)$$

Suppose that $g$satisfies the following two conditions,

(i) For each integer$i$with$1 \leq i \leq k+1$the function $g(z_1, z_2, \dots z_{i-1}, z_{i+1}, z_{k+1})$is weakly monotonic in$z_i$ for fixed$z_1, z_2, \dots z_{i-1}, z_{i+1}, z_{k+1}$.
(ii)If $(m, M)$is a solution of the system

$$m = g(m_1, m_2, \dots, m_{k+1})$$

and

$$M = g(M_1, M_2, \dots, M_{k+1})$$

then $m = M$, for each $i = 1,2, \dots, k+1$, we set,

$$m_i = \begin{cases} mif gisnondecreadinginz_i \\ Mif gisnonincreasinginz_i \end{cases}$$

and

$$M_i = \begin{cases} mif gisnondecreadinginz_i \\ Mif gisnonincreasinginz_i \end{cases}$$

Then $m_i(m, M) = M_i(m, M)$
Then there exists exactly one equilibrium $\bar{x}$ and every solution of (5) converges to $\bar{x}$.

III. PERIODIC SOLUTIONS

In this section, we investigate the periodic character of the positive solutions of (1).
Definition 3.1. (Periodicity)
A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period $p$if $x_{n+p} = x_n$for all $n \geq -k$.
*Theorem 3.2***.** If k is an even positive integer, then (1) has no positive solutions of prime period two for all

$$A, B, \alpha, \beta \in (0, \infty).$$

Proof. Assume for the sake of contradiction that there exists distinctive positive real numbers φ and ψ, such that

$$\dots, \varphi, \psi, \varphi, \psi, \dots$$

is a prime period two solution of (1). If k is even, then $x_n = x_{n-k}$. It follows from the rational difference equation (1) that

$$\varphi = (A + B + C)\psi + \frac{(\alpha + 1)\psi}{\beta + \psi}$$

and

$$\psi = (A + B + C)\varphi + \frac{(\alpha + 1)\varphi}{\beta + \varphi}$$

Simplifying this we get

$$\beta\varphi + \psi\varphi = AB\psi + B\beta\psi + C\beta\psi + A\psi^2 + B\psi^2 + C\psi^2 + \alpha\psi + \alpha$$

and

$$\beta\psi + \psi\varphi = AB\varphi + B\beta\varphi + C\beta\varphi + A\varphi^2 + B\varphi^2 + C\varphi^2 + \alpha\varphi + \alpha$$

By subtracting, we deduce that

$$(\varphi - \psi)\{\beta(A + B + C + 1) + (\varphi + \psi)(A + B + C) + \alpha + 1\} = 0$$

This implies $\varphi = \psi$. This contradicts the hypothesis$\varphi \neq \psi$. Thus, the proof of Theorem 3.2 is completed. Hence the periodic solution of (1) is arrived.

IV. LOCAL STABILITY OF THE EQUILIBRIUM POINT (1)

This section deals with study the local stability character of the equilibrium point of (1). An equilibrium point of (1) is given by

$$\bar{x} = a\bar{x} + \frac{b + c}{\alpha + \beta}$$

If$a < 1$, then the only positive equilibrium point of is given by

$$\bar{x} = \frac{b + c}{(\alpha + \beta)(1 - \alpha)}$$

Let $f: (0, \infty)^4 \rightarrow (0, \infty)$ be a continuously differentiable function defined by

$$f\,(u,v,w,t) = au + \frac{bv + cw + dt}{\alpha v + \beta w + \gamma t} \qquad (6)$$

This follows that

$$\frac{\partial f(u,v,w,t)}{\partial u} = a$$

$$\frac{\partial f(u,v,w,t)}{\partial v} = \frac{(b\beta - c\alpha)w + (b\gamma - d\alpha)t}{(\alpha v + \beta w + \gamma t)^2}$$
$$\frac{\partial f(u,v,w,t)}{\partial w} = \frac{-(b\beta - c\alpha)v + (c\gamma - d\beta)t}{(\alpha v + \beta w + \gamma t)^2}$$
$$\frac{\partial f(u,v,w,t)}{\partial t} = \frac{-(b\gamma - d\alpha)v + (c\gamma - d\beta)t}{(\alpha v + \beta w + \gamma t)^2}$$

Then we get ,

$$\frac{\partial f(\,\bar{x},\bar{x},\bar{x},\bar{x})}{\partial u} = a = -a_3$$

$$\frac{\partial f(\,\bar{x},\bar{x},\bar{x},\bar{x})}{\partial v} = \frac{(b\beta - c\alpha) + (b\gamma - d\alpha)}{(\alpha + \beta + \gamma)^2\bar{x}} = \frac{[(b\beta - c\alpha) + (b\gamma - d\alpha)](1-\alpha)}{(\alpha + \beta + \gamma)(b + c + d)} = -a_2$$

$$\frac{\partial f(\,\bar{x},\bar{x},\bar{x},\bar{x})}{\partial w} = \frac{-(b\beta - c\alpha) + (c\gamma - d\beta)}{(\alpha + \beta + \gamma)^2\bar{x}} = \frac{[-(b\beta - c\alpha) + (c\gamma - d\beta)](1-\alpha)}{(\alpha + \beta + \gamma)(b + c + d)} = -a_1$$

$$\frac{\partial f(\,\bar{x},\bar{x},\bar{x},\bar{x})}{\partial t} = \frac{-(b\gamma - d\alpha) - (c\gamma - d\beta)}{(\alpha + \beta + \gamma)^2\bar{x}} = \frac{[-(b\gamma - d\alpha) - (c\gamma - d\beta)](1-\alpha)}{(\alpha + \beta + \gamma)(b + c + d)} = -a_0$$

Then the linearized equation of (1) about $\bar{x}$is

$$y_{n+1} + a_3 y_n + a_2 y_{n-1} + a_1 y_{n-2} + a_0 y_{n-3} = 0 \qquad (7)$$

whose characteristic equation is

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \qquad (8)$$

Theorem 4.1. Assume that

$$(\alpha + \beta + \gamma)(b + c + d) > max\begin{cases}|2\alpha(c + d) - 2b(\beta + \gamma)|,\\ |2\gamma(b + c) - 2d(\alpha + \beta)|,\\ |2\beta(b + d) - 2c(\alpha + \gamma)|.\end{cases} \quad (9)$$

Then the positive equilibrium point of (1) is locally asymptotically stable.

*Proof. By theorem 2.4, a*ssume that $p_i \in R$, $i = 1, 2, \ldots, k$and $k \in \{0, 1, 2, \ldots\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, n = 0,1,2, \ldots$$

From the above result, it is follows that, (7) is asymptotically stable if all roots of (8) lie in the open disc $|\lambda| < 1$ that is if

$$|a_0| + |a_1| + |a_2| + |a_3| < 1$$

Then,

$$|a|+\left|\frac{[(b\beta-c\alpha)+(b\gamma-d\alpha)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right|$$

$$+\left|\frac{[-(b\beta-c\alpha)+(c\gamma-d\beta)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right|$$

$$+\left|\frac{[-(b\gamma-d\alpha)-(c\gamma-d\beta)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right|<1,$$

also we have,

$$\begin{gathered}|[(b\beta-c\alpha)+(b\gamma-d\alpha)](1-a)| \\ +|[-(b\beta-c\alpha)+(c\gamma-c\alpha)+(c\gamma-d\beta)](1-a)| \\ +|[-(b\gamma-d\alpha)-(c\gamma-d\beta)](1-a)| \\ <[(\alpha+\beta+\gamma)(b+c+d)](1-a).\end{gathered}$$

dividing the denominator and numerator by $(1-a)$, we get

$$\begin{gathered}|(b\beta-c\alpha)+(b\gamma-d\alpha)| \\ |-(b\beta-c\alpha)+(c\gamma-d\beta)| \\ +|-(b\gamma-d\alpha)-(c\gamma-d\beta)|\end{gathered}$$

$<(\alpha+\beta+\gamma)(b+c+d)$ (10)

If

$$\begin{gathered}B_1=(b\beta-c\alpha)+(b\gamma-d\alpha), \\ B_2=-(b\beta-c\alpha)+(c\gamma-d\beta), \\ B_3=-(b\gamma-d\alpha)+(c\gamma-d\beta),\end{gathered}$$

Then we will have the following four cases

Case $(i) B_1>0, B_2>0$ and $B_3>0$.

In this case we see from (10) that

$$\begin{gathered}(b\beta-c\alpha)+(b\gamma-d\alpha)-(b\beta-c\alpha) \\ +(c\gamma-d\beta)-(b\gamma-d\alpha)-(c\gamma-d\beta) \\ <(\alpha+\beta+\gamma)(b+c+d),\end{gathered}$$

If and only if

$$(\alpha+\beta+\gamma)(b+c+d)>0,$$

and hence this is always true.

Case $(ii) B_1<0, B_2>0$ and $B_3>0$.

It follows from (10) that

$$\begin{gathered}(b\beta-c\alpha)+(b\gamma-d\alpha) \\ -(b\beta-c\alpha)+(c\gamma-d\beta) \\ +(b\gamma-d\alpha)+(c\gamma-d\beta) \\ <(\alpha+\beta+\gamma)(b+c+d),\end{gathered}$$

If and only if

$$2\gamma(b+c)-2d(\alpha+\beta)<(\alpha+\beta+\gamma)(b+c+d),$$

which is satisfied by condition(7).

Case $(iii) B_1>0, B_2<0$ and $B_3>0$.

We observe from (10) that,

$$\begin{gathered}(b\beta-c\alpha)+(b\gamma-d\alpha) \\ +(b\beta-c\alpha)-(c\gamma-d\beta) \\ -(b\gamma-d\alpha)-(c\gamma-d\beta) \\ <(\alpha+\beta+\gamma)(b+c+d),\end{gathered}$$

If and only if

$$2\beta(b+d)-2c(\alpha+\gamma)<(\alpha+\beta+\gamma)(b+c+d),$$

which is true.

Case (iv) $B_1>0, B_2>0$ and $B_3<0$.

It follows from (10) that

$$\begin{gathered}(b\beta-c\alpha)+(b\gamma-d\alpha)\\+(b\beta-c\alpha)+(c\gamma-d\beta)\\+(b\gamma-d\alpha)-(c\gamma-d\beta)\\<(\alpha+\beta+\gamma)(b+c+d),\end{gathered}$$

If and only if

$$2b(\beta+\gamma)-2\alpha(c+d)<(\alpha+\beta+\gamma)(b+c+d),$$

Which is satisfied by (9).This is always true. Hence the proof is complete and concludes that (1) is locally asymptotically stable.

V. BOUNDEDNESS CHARACTER OF (1)

In this section, we investigate the boundedness character of the positive solutions of (1).The following theorem is a new result about the boundedness of solutions of (1). *Theorem 5.1*Every solution of (1) is bounded if

$$(i)\gamma<A+B\frac{\beta}{C}+C\frac{\alpha}{B}$$

or

$$(ii)\gamma=A+B\frac{\beta}{C}+C\frac{\alpha}{B}\text{and}\quad \alpha=\beta\gamma=0$$

or

$$(iii)\gamma=A,\qquad \alpha=\beta=0$$

Proof. Assume for the sake of contradiction that (1) has an unbounded solution $\{x_n\}$. Then there exists a subsequence $\{x_{n_i}\}$such that$x_{n_i}\to\infty$and for every i ,

$$x_{n_i}>x_j,\text{ for all } j<n_i+1$$

Now consider,

$$x_{n+1}=Ax_{n_i}+Bx_{n_i-1}+\frac{\alpha x_{n_i-1}+\beta x_{n_i-2}+\gamma x_{n_i-3}}{A+Bx_{n_i-2}+Cx_{n_i-3}}$$

this implies that

$$x_{n_i-2}\to\infty$$

Hence the sequences

$$\{x_{n_i}\},\{x_{n_i-1}\}$$

are completely bounded. In a similar way we conclude that

$$\{x_{n_{i-2}}\},\{x_{n_i-3}\}$$

are also bounded.

Proof of (i). Let ϵ be a positive number such that

$$(B+C)\epsilon<-\gamma+A+B\frac{\beta}{C}+C\frac{\alpha}{B}$$

Eventually,

$$x_{n_i}>\frac{\beta}{C}-\epsilon\text{ and }x_{n_{i-1}}>\frac{\alpha}{B}-\epsilon$$

This provides

$$x_{n_{i+1}}<x_{n_{i-2}}$$

which contradicts the following assumption

$$x_{n_i} > x_j, \text{ for all } j < n_i + 1$$

and completes the proof of (i) and hence

$$x_{n_{i+1}} > x_{n_{i-2}}$$

(ii) Assume that either

$$\alpha = \beta = 0 \text{ and } \gamma > 0$$

or

$$\alpha = \gamma = 0 \text{ and } \beta > 0$$

In this case we have

$$x_{n_i} > \frac{\beta}{C}$$

We have arrived that from (i) that

$$x_{n_{i+1}} > x_{n_{i-2}}$$

and then

$$x_{n_{i-2}} < \frac{\beta x_{n_{-1}}}{A + Bx_{n_i} + Cx_{n_{-1}} - A - B\frac{\beta}{\alpha}} < \frac{\beta}{\alpha}$$

This completes (ii).

$$(iii)\gamma = A, \qquad \alpha = \beta = 0$$

when $A \geq \gamma$ every solution converges to the equilibrium and so is bounded.

*Theorem 5.2*Assume that

$$\frac{\alpha}{B} = \frac{\beta}{C}$$

Then the equilibrium $\bar{x}$ of (1) is a global attractor of all solutions of (1) if and only if,

$$\gamma \leq A + B\frac{\beta}{C} + C\frac{\alpha}{B}$$

Proof.Equation (1) can be written in the normalized form as,

$$x_{n+1} = \frac{\alpha + Bx_n + x_{n-1} + \gamma x_{n-2}}{A + Bx_n + x_{n-1}}, n = 0,1,2, \ . \ . \ . (11)$$

Now it is enough to show that the equilibrium of (1) is a global attractor of all solutions when

$$\gamma \leq A + B + 1$$

We divide the proof into the following three cases,

Case (i)

$$\gamma > A - \alpha$$

We claim that there exists$N$, which is sufficiently large, such that

$$x_n > 1, \text{ for } n \geq N$$

Otherwise for some $N \geq 0$,

$$x_{N+1} = \frac{\alpha + Bx_N + x_{N-1} + \gamma x_{N-2}}{A + Bx_N + x_{N-1}} \leq 1$$

This implies that

$$x_{N-2} < \frac{A - \alpha}{\gamma} < 1$$

in a similar way we have,

$$x_{N-5} < (\frac{A - \alpha}{\gamma})^2$$

This leads to a contradiction. Hence our claim is true. By the change of variables we have

$$y_n = x_n - 1$$

From (11) we have,

$$y_{n+1} = \frac{\alpha - A + \gamma + \gamma y_{n-2}}{A + B + 1 + By_n + y_{n-1}}$$

we established that every solution of the equation above converges to the equilibrium when

$$A + B + 1 \geq \gamma$$

Case (ii)

$$\gamma = A - \alpha$$

Observe that

$$x_{n+1} - 1 = \frac{(A - \alpha)(x_{n-2} - 1)}{A + Bx_n + x_{n-1}}$$

then

$$|x_{n+1} - 1| \leq \frac{A - \alpha}{A} |x_{n-2} - 1|$$

and hence the results.

Case (iii)

$$\gamma < A - \alpha$$

Now we claim that there exists N, which is sufficiently large, such that

$$x_n \leq \frac{A - \alpha}{\gamma}, n \geq N$$

Suppose for the sake of contradiction that for some $N \geq 0$

$$x_{N+1} = \frac{\alpha + Bx_N + x_{N-1} + \gamma x_{N-2}}{A + Bx_N + x_{N-1}} > \frac{A - \alpha}{\gamma}$$

From this we have

$$x_{N-2} > \frac{A}{\gamma} . \frac{A - \alpha}{\gamma}$$

Similarly we have,

$$x_{N-5} > (\frac{A}{\gamma})^2 . \frac{A - \alpha}{\gamma}$$

This leads to a contradiction. Hence our claim is true. Now set

$$S = \lim_{n \to \infty} sup x^n$$

$$I = \lim_{n \to \infty} inf x^n$$

Then clearly,

$$S \leq \frac{\alpha + (B + \gamma + 1)S}{A + (B + 1)S}$$

and

$$I \geq \frac{\alpha + (B + \gamma + 1)I}{A + (B + 1)I}$$

from this we have

$$S = I = \bar{x}$$

and the proof is complete.In all cases the filter was able to converge and accurately track the noise level.

VI. STABILITY OF AN EXTENDED KALMAN FILTER

The extended Kalman filter can be considered as a general state estimator for nonlinear stochastically excited systems in the continuous time case as well as in the discrete time case. It originated from the Kalman filter developed primarily for linear system applications and is an extension of the concept to the nonlinear estimation problem. In the non linear case the estimation of the state vector is corrupted by noise is commonly carried out by a device which is a compromise between accuracy and practical computational complexity. State estimation for non linear deterministic systems without disturbing noise is closely related the zero noise case. This approach leads to the so called extended Kalman filter. In this section we address the filtering problem in case the system dynamics (state and observations) is nonlinear. Consider the non-linear dynamics

$$x_{k+1} = f_k(x_k) + w_k$$
$$y_k = h_k(x_k) + v_k$$

Where,

$$x_k \in \mathbb{R}^n, f_k(x_k): \mathbb{R}^n \to \mathbb{R}^n$$
$$y_k \in \mathbb{R}^n, h_k(x_k): \mathbb{R}^n \to \mathbb{R}^n$$

and $\{v_k\}, \{ w_k \}$ are white Gaussian, independent random processes with zero mean and covariance matrix

$$E[v_k v_k^T] = R_k$$
$$E[w_k e_k^T] = Q_k$$

and x_0 is the system initial condition. The Extended Kalman filter (EKF) gives an approximation of the optimal estimate. The non-linearities of the systems' dynamics are approximated by a linearized version of the non-linear system model around the last state estimate. For this approximation to be valid, this linearization should be a good approximation of the non-linear model in the entire uncertainty domain associated with the state estimate. Extended Kalman filter (EKF) is heuristic for nonlinear filtering problem. We consider the zero-input shift-variant system described by the m^{th} order difference equation with shift variant coefficients

$$y(n) = a_1(n)y(n-1) + a_2(n)y(n-2) + \ldots + a_m(n)y(n-m)$$

where $y(-k), k = 1,2,3,\ldots,m$ are the initial conditions. Recurrence relations can model feedback in a system, where outputs at one time become inputs for future time. Now consider a rational difference equation which is applicable to reduce the noise of various filters like Adaptive, Kalman, Extended Kalman Filter and so on.

$$x_{n+1} = Ax_n + Bx_{n-1} + \frac{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}{A + Bx_{n-2} + Cx_{n-3}}, n = 0,1,2, \ldots$$

Noise is an ever present part of all systems. Any receiver must contend with noise. The Kalman filter removes noise by assuming a pre-defined model of a rational system. Therefore, the Kalman filter model must be meaningful for noise reduction.

Numerical Examples: In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the nonlinear rational difference equation (1).

Example 1. We assumethat $x_{-3} = 3, x_{-2} = 4, x_{-1} = 9, x_0 = 6, A = 0.7,\ B = 8,\ C = 4, \alpha = 4.8,\ \beta = 0.3,$ $\gamma = 2.2$

Plot of $x_{n+1} = Ax_n + Bx_{n-1} + \frac{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}{A + Bx_{n-2} + Cx_{n-3}}$

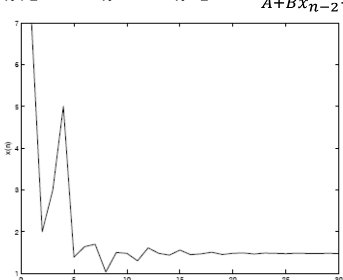

Figure 1 A positive solution of (1) eventually periodic with $x_{-3} = 3, x_{-2} = 4, x_{-1} = 9, x_0 = 6, A = 0.7,\ B = 8,\ C = 4, \alpha = 4.8,\ \beta = 0.3,\ \gamma = 2.2$

Example 2. We assume that$x_{-3} = 4, x_{-2} = 9, x_{-1} = 6, x_0 = 5, A = 2,\ B = 7,\ C = 3, \alpha = 4,\ \beta = 1.3,\ \gamma = 3.2$

Plot of $x_{n+1} = Ax_n + Bx_{n-1} + \frac{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}{A + Bx_{n-2} + Cx_{n-3}}$

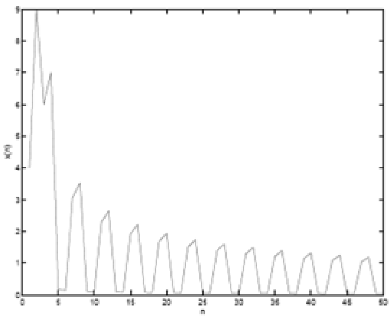

Figure 2 A positive solution of (1) eventually periodic with $x_{-3} = 4, x_{-2} = 9, x_{-1} = 6, x_0 = 5, A = 2,\ B = 7$, $C = 3, \alpha = 4,\ \beta = 1.3,\ \gamma = 3.2$

Example 3. We assume that $x_{-3} = 0.7, x_{-2} = 0.5, x_{-1} = 3, x_0 = 4, A = 7,\ B = 12,\ C = 9, \alpha = 4.8,\ \beta = 3.3,\ \gamma = 9$

Plot of $x_{n+1} = Ax_n + Bx_{n-1} + \frac{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}{A + Bx_{n-2} + Cx_{n-3}}$

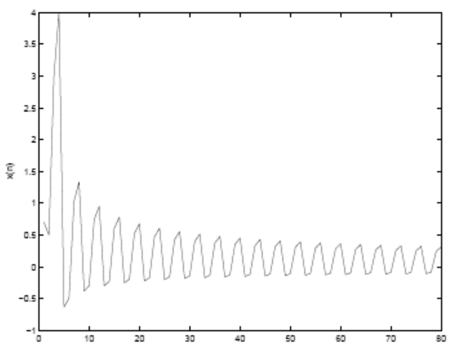

Figure 3 A positive solution of (1) eventually periodic with $x_{-3} = 0.7,\ x_{-2} = 0.5, x_{-1} = 3, x_0 = 4,\ A = 7,\ B = 12$, $C = 9, \alpha = 4.8,\ \beta = 3.3,\ \gamma = 9$

Example 4. We assume that $x_{-3} = 7, x_{-2} = 11, x_{-1} = 0.3, x_0 = 4, A = 6,\ B = 11,\ C = 9, \alpha = 4,\ \beta = 3,\ \gamma = 7$

Plot of $x_{n+1} = Ax_n + Bx_{n-1} + \frac{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}{A + Bx_{n-2} + Cx_{n-3}}$

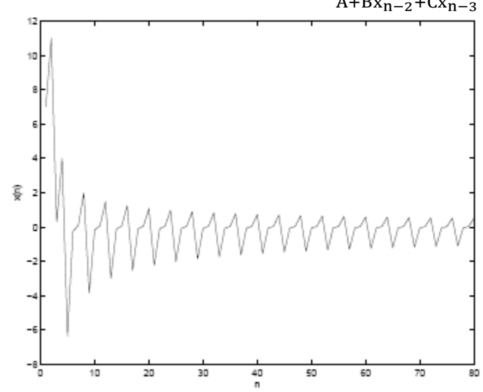

Figure 4 A positive solution of (1) eventually periodic with $x_{-3} = 7,\ x_{-2} = 11, x_{-1} = 0.3, x_0 = 4,\ A = 6,\ B = 11$, $C = 9, \alpha = 4,\ \beta = 3,\ \gamma = 7$

VII. CONCLUSION

We have investigated the periodic solutions, exponential stability and boundedness characters of (1). The results we obtained are implied in an extended Kalman filter for noise reduction. This allows for lower estimation error when used a system of rational difference equations. Simulation results are arrived through various numerical examples to reduce the noise level.

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